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Kelvin's discovery of Taylor columns

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ABSTRACT

In 1868 Kelvin observed that a sphere moving on the axis of uniformly rotating water takes with it a column of liquid as if this were a rigid mass. This observation anticipates by thirty years Hough's analytical demonstration that steady slow motion in a rotating fluid is two-dimensional, and by fifty years Taylor's perception and experimental demonstration of the consequences thereof. This note discusses Kelvin's unpublished observation in the context of his lifelong interest on rotatory motion and later investigations on the subject by Hough, Proudman, Taylor, and others.

Contents

1. Introduction

The Taylor–Proudman theorem was first obtained by Hough [1] in a study of tides and ocean currents and it was later rediscovered by Proudman [2] and Taylor [3] in studies of the motion of solids through fluids possessing vorticity. It establishes that slow motions superposed on a fluid already in solid-body rotation are two-dimensional, in the sense that material lines parallel to the axis of rotation remain so and material particles preserve their spacings along these lines. The theorem is generally deduced using the equations of motion in a rotating frame of reference, but it can also be deduced in the non-rotating frame using Kelvin circulation theorem [4] or the vorticity equation first proved by Helmholtz [5]: $D\vec{\omega}/Dt = \vec{\omega} \cdot \nabla\vec{u}$. Under the assumption of slow motion superposed on solid-body rotation, the vorticity can be considered as composed of two parts: one is due to the uniform rotation and the other is due to the slow motion. If the latter is neglected compared to the former then vorticity is preserved by material particles and, according to Helmholtz' formula, $\vec{\omega} \cdot \nabla\vec{u} = 0$. Consequently, the velocity does not change in the direction of the axis of rotation and there is neither tilting nor stretching of material lines parallel to this axis. The motion is thus two-dimensional in the sense defined above.

Taylor [6] perceived a most remarkable consequence of the Taylor–Proudman theorem, namely, that a three-dimensional body moving slowly through a fluid which is in uniform rotation takes with it the fluid bounded by the straight cylinder whose generators are tangent to the body and parallel to the axis of rotation. He also demonstrated this striking property experimentally: with a sphere moving along the axis of rotation [7] and with a cylinder, shorter than the water height, moving perpendicularly to the axis of rotation [6]. In recognition to this, the fluid within the bounding cylinder is now called a "Taylor column".

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The purpose of this note is to present Kelvin's[1] earlier observation of this phenomenon, as described in a letter addressed to him by James Thomson, his elder brother, on 30 September 1868 [8]. Although Kelvin [9,10] discussed similar experiments in one of his papers and demonstrated them during lectures, James Thomson's letter is, to the best of my knowledge, the only written account of Kelvin's observation of a Taylor column.

2. Kelvin and rotating motion of solids and fluids

Rotating motion, in solids and fluids, attracted Kelvin's attention throughout his life. As a sixteen-year-old undergraduate, Kelvin was already acquainted with the theory of rotating bodies, as it is demonstrated by his "Essay on the figure of the Earth" which deserved the class prize in astronomy for the year 1840 at the University of Glasgow [11, p. 10]. Four years later, while staying at the coastal town of Cromer to prepare for the mathematical tripos examination at Cambridge, he wrote to his father: "I have been investigating, as far as I have been able in the time I have been able to spare, the theory of spinning tops and rolling hoops, which is very curious and difficult. Blackburn[2] and I have been making a great many experiments on the subject, and have collected quite a cabinet of ellipsoids of various proportions, which we find on the beach, besides having got a teetotum, humming top, and peery. Some of the results we have obtained are very curious..." [11, p. 82]. According to Perry [12, p. 68] one of the results obtained by Kelvin and Blackburn at this time was the mathematical explanation of the rising of a spinning top, thus anticipating the solutions of A. Smith and J.H. Jellett.

In the 1850s the dynamics of rotating bodies became the subject of widespread interest. The theoretical advances of the previous one hundred years motivated the development of experimental means to verify or demonstrate them. Thus, throughout the nineteenth century several investigators constructed devices consisting of a spinning mass whose axis of rotation is free to take any orientation. Examples are Bohnenberger's *Maschine* in Germany in the 1810s and Johnson's *rotascope* in the United States in the 1830s. But it was only after Foucault used his *gyroscope* to exhibit the rotation of the Earth in 1852 that the device acquired a popularity that extended well beyond academic circles. In the 1860s Kelvin designed his own apparatus: it was essentially a gyroscope with the mass concealed in a case so that its rotation would not be visible. Kelvin [13] called it *gyrostat*, to stress the fact that rotating motion confers stability.

In 1856, while analysing magneto-optical phenomena, Kelvin [14] wondered "whether all matter is continuous, and molecular heterogeneousness consists in finite vortical or other relative motions of contiguous parts of a body", but cautiously added that it was "perhaps in vain to speculate, in the present state of the science". This notwithstanding, in January 1858 Kelvin continued thinking about the possibility that all phenomena of matter could be explained "if it were possible to conceive the properties of one particular substance to be owing to a particular form & order of motions or eddies in a fluid, and to remain as constant as they do in nature through all combinations and actions of all kinds to which they may be subjected" [15]. Kelvin attributed three essential properties to matter's ultimate constituting elements (he initially disliked the word atom because, at that time, it denoted an infinitely-hard, solid particle). Those properties were permanence, elasticity and capacity to act at a distance.

A few months later Helmholtz [5] published his seminal memoir on vortex motion where he demonstrated "the absolute permanence of the rotation, and the unchangeable relation (...) between it and the portion of the fluid once acquiring such motion in a perfect fluid", as Kelvin would summarize Helmholtz laws of vortex motion nine years later [11, p. 514]. Helmholtz also demonstrated that filamentary vortices act at a distance: he fully described the mutual influence, and ensuing evolution, of two parallel line vortices, now known as point vortices; as well as the interaction of two circular, coaxial vortex rings. Helmholtz' memoir contained no hint as to the possibility of vortex filaments possessing some form of elasticity. Thus when Kelvin first read it, in late 1858 or early 1859 [11, p. 402], it did not occurred to him that the theory of vortex motion could be of use as a framework for building up a theory of matter. Kelvin, however, could have foreseen it. For he had noticed, as early as 1847, that rotation confers elasticity to a rotating liquid, as he described in a letter to Stokes: "I perceived a fine instance of elasticity in an incompressible liquid, in a very simple observation made at Paris, on a cup of thick "chocolat au lait". When I made the liquid revolve in the cup, by stirring it, and then took out the spoon, the twisting motion (...) in becoming effaced, always gave rise to several oscillations so that before the liquid began to move as a rigid body, it performed oscillations like an elastic (incompressible) solid" [16, p. 30]. Anyhow, it was not until January 1867, when he saw a demonstration of smoke rings by Peter Guthrie Tait, that Kelvin perceived the elasticity of tubular vortices. He immediately realised that Helmholtz' theory of vortex motion possessed the necessary elements to build his long-sought-after theory of matter.

In February Kelvin [17] read before the Royal Society of Edinburgh a paper "On vortex atoms", where he presented his research programme for the construction of such a theory. He also announced the solution for waves of deformation of the cross section of a cylindrical vortex, the proof of which he would publish thirteen years later [18]. In April he read a paper "On vortex motion", which had the explicit objective of advancing his vortex atom hypothesis. Most of the paper is devoted to motion of solids in an incompressible frictionless fluid, but the main result is the introduction of the concept of circulation and the theorem stating that this is conserved by any material circuit. A year later Kelvin [19] recast and enlarged this memoir and continued his work on the transverse vibrations of a straight vortex [20].

3. Kelvin's observation of a Taylor column

On 30 September 1868 James Thomson wrote Kelvin a letter, reproduced below, in which he describes Kelvin's observation of what is now known as a Taylor column. The letter contains a simple diagram illustrating the parabolic free surface of a liquid in uniform rotation; our figures 1a, b, c (referenced here between square brackets) are drawn after Kelvin [9] and the descriptions of Thomson [20] and Perry [12]. The omitted paragraphs contain references to James Thomson's work on the potential vortex (*whirlpool of free mobility*), including a long quotation from one of his papers [21].

12 Lansdown Road
Dublin. 30 Sept. 1868.

My dear William,

I saw Prof. Tait in Belfast last week and he mentioned to me an experiment which you, or you and he, had been making in which you put a cork into the axis of a whirlpool of water having the same angular velocity at all parts, and got the result that the cork stayed down without floating up to the top [fig. 1a]; and he mentioned that if you put in two corks at dif-

[1] William Thomson became *Baron Kelvin of Largs* in 1892, when a peerage was conferred upon him. Therefore throughout the text we will refer to him as Kelvin but we will list his papers under *Thomson, W.*, since this is the name that appears on the papers discussed here.

[2] Hugh Blackburn was a lifelong friend of Kelvin. He became Professor of Mathematics at Glasgow University in 1849.

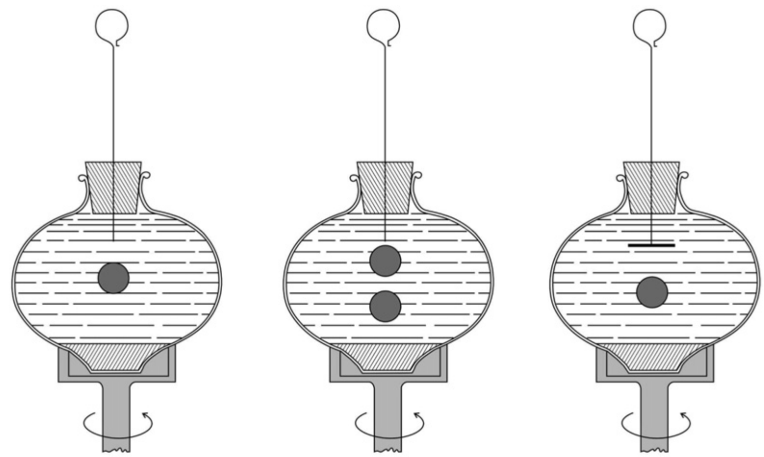

Fig. 1. (a) Kelvin's experiment to show "elasticity" acquired by a rotating liquid, first published in [9]. (b) Kelvin's observation of a Taylor column as described by Thomson [21]. (c) The variation of G.F. Fitzgerald as described by Perry [12].

ferent levels in the axis, they would both stay where you put them [fig. 1b], and that if you pushed the upper one down a little the lower one would go down too, as if there was an almost rigid mass of water between them. I think in the above I have correctly described the main points he told me of the experiments.

He asked me whether I knew anything of such matters before. I told him that I had long ago perceived and told people (my own students I believe and probably yourself and I think I knew it long before I was a Professor[3] and discussed it in my office) that in a whirlpool of equal angular velocities at all parts such as would have a parabola as the vertical diametrical section of its upper surface a duck could not swim from one part to a higher or lower part without having to expend mechanical energy of a real amount, not exceedingly small: while a duck on a still pond with level surface could swim from place to place at a slow rate by the expenditure of very little energy and infinitely slowly by the expenditure of only an infinitely small amount of energy.– also that in the whirlpool to which I gave the name the Whirlpool of Free Mobility, the duck could swim from place to place without expenditure of energy in like manner as in still water:– Also that a fish in the interior of any whirlpool, except the one of Free mobility, would require to give out a real finite quantity of energy in order to move from place to place throughout the mass of water (except of course along some particular curved faces or surfaces in the interior of the fluid as you can easily perceive).

[. . .]

Prof. Tait asked me to let you know about this because he says the published name "The Whirlpool of Free Mobility" gives full proof or indication that I understood at the time of the paper that no other whirlpool could have free mobility and he says that you are publishing something on the subject and he believes you would like to have the things done by me which I have alluded to brought under your notice at present.

Your affte brother

J. Thomson

Kelvin demonstrated some of the experiments described in this letter during a lecture bearing the suggestive title "Elasticity viewed as possibly a mode of motion" which he gave before the Royal Institution of Great Britain on 4 March 1881. The published summary of this lecture describes the experiments as follows [10]: "A little wooden ball, which when thrust down under still water jumped up again in a moment, remained down as if embedded in jelly when the water was caused to rotate rapidly, and sprang back, as if the water had elasticity like that of jelly, when it was struck by a stiff wire pushed down through the centre of the cork by which the glass vessel containing the water was filled." Kelvin [9] quoted this passage, and illustrated it with the diagram reproduced in figure 1a, in a footnote of his paper "On the stability of steady and of periodic fluid motion" [9].

John Perry, a student of Kelvin in the 1870s, exhibited this and other examples of quasi-rigidity conferred by rotation during a lecture before the British Association for the Advancement of Science on 6 September 1890. The published account of this lecture describes the experiment as follows [12, p. 17]: "The water inside this glass vessel [figure 1c] is in a state of rapid motion, revolving with the vessel itself. Now observe a piece of paraffin wax A immersed in the water, and you will see when I push at it with a rod that it vibrates just as if it were surrounded with a thick jelly. Let us now apply Prof. Fitzgerald's[4] improvement on this experiment of Sir William Thomson's. Here is a disc B stuck on the end of the rod; observe that when I introduce it, although it does not touch A, A is repelled from the disc".

4. Final comments

From the 1850s on Kelvin studied rotating motion primarily as a means to construct a theory of matter (see, e.g., [15,22]). When he finally abandoned this idea, around 1900, he characterised his efforts as a failure. Kelvin certainly did not reach his goal but while pursuing it he produced profound insights into vortex dynamics. The most important are the circulation theorem and the modes of oscillation of a straight tubular vortex, both of which now bear his name. It was in the period of most intense work on these subjects that Kelvin made the experiments discussed in this note. One can only speculate as to the reasons why he did not pursue this subject further, for at least two observed facts called for an explanation: (a) the buoyant solid did not float and (b) a column of liquid moved with the solid sphere. Kelvin [23–25] published extensively on the motion of solids immersed in an irrotational flow

[3] James Thomson became Professor of Engineering at Queen's College, Belfast, in 1857.

[4] George Francis FitzGerald, Professor of Natural and Experimental Philosophy at Trinity College, Dublin, discovered the FitzGerald–Lorentz contraction.

but he did not publish any results on the motion of solids immersed in a rotational flow. This is precisely the subject that would be treated forty five years later by Proudman [2] and Taylor [3].

The ultimate explanation of the formation of the liquid column in front of the moving sphere is, of course, the Taylor–Proudman theorem [1–3]. But, since this applies to a perfect fluid only, the flow around a sphere moving along the axis of a rotating, slightly-viscous fluid was extensively studied during the XX century, both experimentally [7,26,27] and theoretically [7,28,29].

It is the observation of a buoyant sphere not floating to the surface that still remains unexplained. Fultz [30] suggests that the sphere did move, but very slowly (Maxworthy [26], for instance, measured speeds of a fraction of a millimetre per second in a fluid rotating at about 160 rpm). Even this slow motion, however, should become apparent after a few seconds. Yet Thomson [8], Kelvin [10] and Perry [12] reported a steady sphere. Thus a likelier explanation is that the buoyancy force on the sphere is balanced by a static drag produced by the rotating fluid contained in a small vessel. For as the length of the container decreases, the influence of the bottom and top boundaries on the front and rear Taylor columns greatly increases, which results in an increased drag coefficient [26]. The fact that a buoyant sphere has not been observed to remain steady within a rapidly rotating fluid, either in laboratory experiments [26] or in numerical simulations [31], may simply mean that the parameter values in Kelvin's experiment lie outside the range covered in later studies.

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